

# ANOMALY FORMULAS FOR THE COMPLEX-VALUED ANALYTIC TORSION ON COMPACT BORDISMS

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**ABSTRACT.** We extend the complex-valued analytic torsion, introduced by Burghelea and Haller on closed manifolds, to compact Riemannian bordisms. We do that by studying non-selfadjoint Laplacians and considering absolute and relative boundary conditions. In this context, we obtain anomaly formulas for the complex-valued analytic torsion based on the work by Brüning and Ma for the Ray–Singer metric on manifolds with boundary. In odd dimensions, our anomaly formulas are in accord with the corresponding results of Su, without requiring the variations of the Riemannian metric and bilinear structures to be supported in the interior of the manifold. Our results are proved by using invariance theory for the coefficients of the heat kernel asymptotic expansion and an argument of analytic continuation.

## INTRODUCTION

In this paper, we denote by  $(M, \partial_+ M, \partial_- M)$  a compact Riemannian bordism. That is,  $M$  is a compact Riemannian manifold of dimension  $m$ , with Riemannian metric  $g$ , whose boundary  $\partial M$  is the disjoint union of two closed submanifolds  $\partial_+ M$  and  $\partial_- M$ . For  $E$  a flat complex vector bundle over  $M$ , we consider generalized Laplacians acting on the space  $\Omega(M; E)$  of  $E$ -valued smooth differential forms on  $M$  satisfying absolute boundary conditions on  $\partial_+ M$  and relative boundary conditions on  $\partial_- M$ .

We study the *complex-valued* Ray–Singer torsion on  $(M, \partial_+ M, \partial_- M)$ . This torsion was introduced by Burghelea and Haller on closed manifolds, see [BH07] and [BH10], as a complex-valued version for the Ray–Singer torsion or *metric*, originally studied by Ray and Singer in [RS] for unitary flat vector bundles on closed manifolds. Our main result, Theorem 3, exhibits a logarithmic derivative of the complex-valued analytic torsion on compact Riemannian bordisms and its proof is based on the work of Brüning and Ma in [BM06] for the Ray–Singer metric on manifolds with boundary.

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To define the complex-valued analytic Ray–Singer torsion, we assume  $E$  admits a fiberwise nondegenerate symmetric bilinear form  $b$  and we proceed as in the closed situation, see [BH07]. The bilinear form, together with the Riemannian metric on  $M$ , induce a nondegenerate symmetric bilinear form on  $\Omega(M; E)$ . We study generalized Laplacians  $\Delta_{E,g,b}$ , also referred as *bilinear* Laplacians, which are formally symmetric, with respect to the bilinear product, on the space of smooth forms satisfying the boundary conditions specified above. Instead, the construction of the Ray–Singer metric needs a Hermitian structure and a selfadjoint Laplacian, which is referred as the *Hermitian* Laplacian, see [RS], [Lü], [Mü78] and [BZ] for instance. The analytic Ray–Singer metric on manifolds with boundary has been studied by several authors, see for instance [RS], [Mü78], [Mü93], [Lü], [DF], [BM06], [BM11] and references therein. In particular, we are interested in the work of Brüning and Ma in [BM06], where the variation of the analytic Ray–Singer torsion, with respect to smooth variations on the underlying Riemannian and Hermitian metrics, was computed; these formulas will be called *anomaly formulas* for short.

In section 1 we use well known theory on boundary value problems for differential operators to treat ellipticity, regularity and spectral properties for the bilinear Laplacian. In particular, under the specified elliptic boundary conditions,  $\Delta_{E,g,b}$  extends to a not necessarily selfadjoint closed unbounded operator in the  $L^2$ -norm, it has compact resolvent and discrete spectrum, all its eigenvalues are of finite multiplicity, its (generalized) eigenspaces contain smooth differential forms only and the restriction of the bilinear form  $\beta_{g,b}$  to each of these is also non degenerate. Proposition 2 gives Hodge decomposition results in the bilinear setting, which are analog to the Hermitian one, see for instance [Mü78], [Lü] and more recently in [BM11]. This section ends with Proposition 3 stating that the 0-generalized eigenspace of  $\Delta_{E,g,b}$  *still* computes relative cohomology  $H(M, \partial_- M; E)$ , without necessarily being isomorphic to it.

In section 2, we recall generalities about the coefficients of the heat kernel asymptotic expansion for an elliptic boundary value problem. These coefficients are spectral invariants and locally computable as polynomial functions in the jets of the symbols of the operators under consideration, see [Se67] and [Se69]. By Weyl’s first Theorem of invariant theory, these coefficients are expressed as universal polynomial in terms of locally computable geometric invariants; these facts, studied in detail by Gilkey in [Gi84] and [Gi04], provide the key ingredients in the proof of Theorem 2, leading to Theorem 3. In [BM06], Brüning and Ma studied the heat kernel for Hermitian Laplacians on manifolds with boundary under absolute boundary conditions and they obtained anomaly formulas for the corresponding Ray–Singer metric. We use

Poincaré duality, see Lemma 7, to infer the corresponding quantities for the Hermitian Laplacian with the boundary conditions specified on the bordism  $(M, \partial_+ M, \partial_- M)$ , see Proposition 5 and Theorem 1. Moreover, we exhibit the holomorphic dependence of the coefficients of the heat kernel asymptotic expansion for these boundary value problems on a complex parameter, this is Lemma 11. An analytic continuation argument then allows to deduce the infinitesimal variation of these quantities for the bilinear case out from those corresponding in the Hermitian situation, see Theorem 2.

In section 3, we define the complex-valued analytic torsion on the compact Riemannian bordisms. Following the approach in [BH07], we obtain a nondegenerate bilinear form on the determinant line  $\det(H(M, \partial_- M; E))$ , denoted by  $\tau_{E,g,b}(0)$  and induced by the restriction of  $\beta_{g,b}$  to the 0-generalized eigenspace of  $\Delta_{E,g,b}$ . The (inverse square of) the complex-valued Ray–Singer torsion for manifolds with boundary is

$$\tau_{E,g,b}^{\text{RS}} := \tau_{E,g,b}(0) \cdot \prod_p (\det'(\Delta_{E,g,b,p}))^{(-1)^p p},$$

where the product above is, in this situation, a non zero complex number with  $\det'(\Delta_{E,g,b,p})$  being the  $\zeta$ -regularized product of all non-zero eigenvalues of  $\Delta_{E,g,b,p}$ . For closed manifolds, the variation of the complex analytic Ray–Singer torsion with respect to smooth changes on the metric  $g$  and the bilinear form  $b$  has been obtained in [BH07]. They did so, by computing the leading and subleading terms in the asymptotic expansion of the heat kernels associated with a certain class of Dirac operators, sections 7 and 8 in [BH07]. They eventually obtained a geometric invariant by introducing appropriate correction terms, see Theorem 4.2 in [BH07]. In [Su], by using techniques from [SZ08], [Ve] and [Mü78], Su generalized the complex-valued analytic Ray–Singer torsion to the situation in which  $\partial_+ M \neq \emptyset$  (or  $\partial_- M \neq \emptyset$ ). Also in that paper, Su proved that in odd dimensions, the complex-valued analytic torsion does not depend neither on smooth variations of the Riemannian metric nor on smooth variations of the bilinear form, as long as these are compactly supported in the interior of  $M$ . This section ends with the statement of Theorem 3, which gives formulas for the variation of the complex-valued analytic Ray–Singer torsion with respect to smooth variations of the metric and the bilinear form; we call such formulas anomaly formulas for the complex-valued analytic torsion. As in the closed situation, see [BH07], the anomaly formulas for the complex-valued Ray–Singer torsion are found by using the results for the constant coefficients of the corresponding heat kernel in section 2.

The anomaly formulas for the Ray–Singer metric in Theorem 1 were also obtained by Bruning and Ma in their recent work on the gluing formulas for the Ray–Singer metric [BM11] continuing their work in [BM06].

The anomaly formulas for the complex-valued analytic torsion stated in Theorem 3 generalize the ones obtained by Burghelea and Haller for the closed situation in [BH07], and also the ones in [Su] by Su in odd dimensions: they do not longer require  $g$  and  $b$  to be constant in a neighborhood of the boundary and both kind of boundary conditions are considered at the same time.

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## 1. BILINEAR LAPLACIANS AND HODGE DECOMPOSITION ON BORDISMS

Let  $(M, \partial_+ M, \partial_- M)$  be a compact Riemannian bordism of dimension  $m$ , by which we mean  $M$  to be a compact connected not necessarily orientable smooth manifold of dimension  $m$  with Riemannian metric  $g$ , whose boundary  $\partial M$  is the disjoint union of two closed submanifolds  $\partial_+ M$  and  $\partial_- M$ . The boundary inherits the Riemannian metric from  $M$ . We do not require any condition on the metric near the boundary. We denote by  $\varsigma_{\text{in}}$  the *geodesic unit inwards pointing normal vector field* on the boundary. Let us denote by  $\Theta_M$  and by  $\Theta_{\partial M}$  the orientation bundles of  $M$  and  $\partial M$  respectively, endowed with the unique flat connections specified by the de-Rham differential on (twisted) forms. For  $i : \partial M \hookrightarrow M$  the canonical embedding, we denote by  $\Theta_M|_{\partial M} = i^* \Theta_M$  the restriction of  $\Theta_M$  to  $\partial M$ ; as real line bundles over  $\partial M$ ,  $\Theta_M|_{\partial M}$  and  $\Theta_{\partial M}$  are identified with the standard convention. If  $TM$  denotes the tangent bundle of  $M$  and  $T\partial M$  the tangent bundle of  $\partial M$ , then the associated Levi-Civita connection on  $TM$  is denoted by  $\nabla$  and that on  $T\partial M$  by  $\nabla^\partial$ . If  $\text{vol}_g(M) \in \Omega^m(M; \Theta_M)$  is the volume form of  $M$ , then for each integer  $0 \leq k \leq m$ , the Hodge  $\star$ -operator is the linear isomorphism  $\star_k := \star_{g,k} : \Omega^k(M) \rightarrow \Omega^{m-k}(M; \Theta_M)$ , defined by  $\alpha \wedge \star \alpha' = \langle \alpha, \alpha' \rangle_g \text{vol}_g(M)$ , where  $\alpha, \alpha' \in \Omega^k(M)$ . Let  $E$  be a flat complex vector bundle with flat connection  $\nabla^E$  and denote by  $\Omega(M; E)$  the space of  $E$ -valued smooth differential forms on  $M$ , together with de-Rahm differential  $d_E := d_{\nabla^E}$ . The flat complex vector bundle dual to  $E$  is denoted by  $E'$  endowed with the induced flat connection  $\nabla^{E'}$ . The completion of  $\Omega(M; E)$  with respect to the  $L^2$ -norm is denoted by  $L^2(M; E)$ .

**1.1. Generalized Laplacians on manifolds with boundary.** We proceed as in [BH07]. Assume  $E$  is endowed with a fiber-wise non degenerate symmetric bilinear form  $b$ . Then, a nondegenerate symmetric bilinear form on  $\Omega(M; E)$  is defined as  $\beta_{g,b}(v, w) := \int_M \text{Tr}(v \wedge \star_b w)$ , where  $\text{Tr} : \Omega(M, E \otimes E' \otimes \Theta_M) \rightarrow \Omega(M; \Theta_M)$  is the trace map induced by the canonical pairing between  $E$  and  $E'$  and the operator  $\star_{b,q} := \star_q \otimes b :$

$\Omega^q(M; E) \rightarrow \Omega^{m-q}(M; E' \otimes \Theta_M)$  is defined by using the Hodge  $\star$ -star operator  $\star_q$  and the isomorphism of vector bundles  $b : E \rightarrow E'$  specified by the bilinear form and also denoted by the same symbol. Thus, the map  $d_{E,g,b,q}^\sharp : \Omega^q(M; E) \rightarrow \Omega^{q-1}(M; E)$  is defined by

$$(1) \quad d_{E,g,b,q}^\sharp := (-1)^q \star_{b,m-(q-1)}^{-1} d_{E' \otimes \Theta_M, m-q} \star_{b,q},$$

where  $d_{E' \otimes \Theta_M}$  is the de-Rham differential on  $\Omega(M; E' \otimes \Theta_M)$  induced by the dual connection on  $E'$ , is a codifferential on  $\Omega(M; E)$ . In this way, the operator

$$(2) \quad \Delta_{E,g,b,q} := d_{E,g-1} d_{E,g,b,q}^\sharp + d_{E,g,b,q+1}^\sharp d_{E,q} : \Omega^q(M; E) \rightarrow \Omega^q(M; E),$$

is a generalized Laplacian in the sense that its principal symbol is a scalar positive real number; that is, the operator  $\Delta_{E,g,b}$  is elliptic and it will be called the *bilinear Laplacian*. A straightforward use of Stokes' Theorem leads to the following Green's formulas; that is,

$$\begin{aligned} \beta_{g,b}(d_E v, w) - \beta_{g,b}(v, d_{E,g,b}^\sharp w) &= \int_{\partial M} i^*(\text{Tr}(v \wedge \star_b w)), \\ (3) \quad \beta_{g,b}(\Delta_E v, w) - \beta_{g,b}(v, \Delta_E w) &= \int_{\partial M} i^*(\text{Tr}(d_{E,g,b}^\sharp v \wedge \star_b w)) - \int_{\partial M} i^*(\text{Tr}(w \wedge \star_b d_E v)) \\ &\quad - \int_{\partial M} i^*(\text{Tr}(d_{E,g,b}^\sharp w \wedge \star_b v)) + \int_{\partial M} i^*(\text{Tr}(v \wedge \star_b d_E w)). \end{aligned}$$

for  $v, w \in \Omega(M; E)$ . To study analytic and spectral properties of the bilinear Laplacian on  $(M, \partial_+ M, \partial_- M)$ , we impose elliptic boundary conditions. The space of smooth forms  $w \in \Omega(M; E)$  satisfying *relative boundary conditions* on  $\partial_- M$  and *absolute boundary conditions* on  $\partial_+ M$  is

$$(4) \quad \Omega(M; E)|_{\mathcal{B}} := \left\{ w \in \Omega(M; E) \left| \begin{array}{ll} i_+^* \star_b w = 0, & i_-^* w = 0 \\ i_+^* d_{E' \otimes \Theta_M, g, b}^\sharp \star_b w = 0, & i_-^* d_{E, g, b}^\sharp w = 0 \end{array} \right. \right\}.$$

Remark that the integrands on the right of formulas in (3) vanish, on forms in  $\Omega(M; E)|_{\mathcal{B}}$ . The operator  $\Delta_{E,g,b}$  with domain of definition  $\Omega(M; E)|_{\mathcal{B}}$  does not extend to a selfadjoint operator on  $L^2(M; E)$ . However, the specified boundary conditions are an example of a more general type called mixed boundary conditions, see [Gi04], which provide elliptic boundary conditions for operators of Laplace type. As in the closed situation, we will see that the spectrum of the bilinear Laplacian possesses properties close to those of a Hermitian Laplacian, see section 1.5.

**1.2. Some Notation.** Let  $i_\pm : \partial_\pm M \hookrightarrow M$  be the canonical embedding of  $\partial_\pm M$  into  $M$  and  $E_\pm := i_\pm^* E$  be the corresponding restriction bundles. For

$1 \leq q \leq m$ , consider the boundary operators

$$\begin{aligned}\mathcal{B}_- : \Omega^q(M; E) &\longrightarrow \Omega^q(\partial_- M; E_-) \oplus \Omega^{q-1}(\partial_- M; E_-), \\ \mathcal{B}_+ : \Omega^q(M; E) &\longrightarrow \Omega^{m-q}(\partial_+ M; E_+) \oplus \Omega^{m-(q-1)}(\partial_+ M; E_+),\end{aligned}$$

respectively defined by

$$(5) \quad \begin{aligned}\mathcal{B}_- w &:= (\mathcal{B}_-^0 w, \mathcal{B}_-^1 w) := (i_-^* w, i_-^* d_{E,g,b}^\# w), \\ \mathcal{B}_+ w &:= (\mathcal{B}_+^0 w, \mathcal{B}_+^1 w) := (\star_b^{\partial M-1} (i_+^* \star_b w), \star_b^{\partial M-1} (i_+^* d_{E' \otimes \Theta_{M,g,b'}}^\# \star_b w))\end{aligned}$$

and

$$\begin{aligned}\mathcal{B} : \Omega^q(M; E) &\longrightarrow \Omega^{m-q}(\partial_+ M; E_+) \oplus \Omega^{m-q-1}(\partial_+ M; E_+) \oplus \Omega^q(\partial_- M; E_-) \oplus \Omega^{q-1}(\partial_- M; E_-) \\ w &\mapsto \mathcal{B}w := (\mathcal{B}_+ w, \mathcal{B}_- w).\end{aligned}$$

Remark that  $w \in \Omega(M; E)_\mathcal{B}$  if and only if  $\mathcal{B}w = 0$ .

**Lemma 1.** *For a subspace  $X \subseteq \Omega(M; E)$ , denote by  $X|_\mathfrak{B} := \{w \in X | \mathfrak{B}w = 0\}$  the space of smooth forms in  $X$  which satisfy the boundary conditions specified by the vanishing of the operator  $\mathfrak{B} \in \{\mathcal{B}_\pm^0, \mathcal{B}_\pm^1, \mathcal{B}_\pm, \mathcal{B}\}$ . Set*

$$(6) \quad X|_{\mathcal{B}^0} := X|_{\mathcal{B}_-^0} \cap X|_{\mathcal{B}_+^0}.$$

Then the following assertions hold

- (a)  $X|_\mathcal{B} = X|_{\mathcal{B}^0} \cap X|_{\mathcal{B}_-^1} \cap X|_{\mathcal{B}_+^1}$  and  $X|_\mathcal{B} \subset X|_{\mathcal{B}^0} \subset X|_{\mathcal{B}_-^0}$ ,
- (b)  $d_E(\Omega(M; E)|_{\mathcal{B}^0}) \subset \Omega(M; E)|_{\mathcal{B}_-^0}$ ,
- (c)  $d_E(\Omega(M; E)|_\mathcal{B}) \subset \Omega(M; E)|_{\mathcal{B}^0}$  and  $d_{E,g,b}^\#(\Omega(M; E)|_\mathcal{B}) \subset \Omega(M; E)|_{\mathcal{B}^0}$ ,
- (d) If  $v \in \Omega(M; E)|_{\mathcal{B}_-^0}$  and  $w \in \Omega(M; E)|_\mathcal{B}$  then  $\beta_{g,b}(d_E v, d_{E,g,b}^\# w) = 0$ ,
- (e) If  $v, w \in \Omega(M; E)|_{\mathcal{B}^0}$ , then  $\beta_{g,b}(d_E v, w) = \beta_{g,b}(v, d_{E,g,b}^\# w)$ ,
- (f) If  $v, w \in \Omega(M; E)|_\mathcal{B}$ , then  $\beta_{g,b}(\Delta_{E,g,b} v, w) = \beta_{g,b}(v, \Delta_{E,g,b} w)$ .

*Proof.* The first assertion is obvious. The remaining assertions follow from (6), (4), the Green's formulas in (3) and straightforward manipulations coming from the definition of the operators and spaces above.  $\square$

**1.3. Boundary conditions and Poincaré duality.** The boundary value problem specified by the operator  $\Delta_{E,g,b}$  acting on  $\Omega(M; E)_\mathcal{B}$  will be denoted by  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E,g,b}$ . This boundary value problem is naturally related to its *Poincaré dual*, by means of the isomorphism  $\star_b$ . To illustrate that, consider the flat complex vector bundle  $E'$  dual to  $E$ , with the dual connection  $\nabla^{E'}$  and dual bilinear form  $b'$ , together on the bordism  $(M, \partial_+ M, \partial_- M)' := (M, \partial_- M, \partial_+ M)$  seen as the *dual bordism* to  $(M, \partial_+ M, \partial_- M)$ . Now, by definition of these operators, the equality  $\star_b d_{E,g,b}^\# d_E = d_{E'} d_{E' \otimes \Theta_{M,g,b'}}^\# \star_b$  holds and therefore

$$\star_b \Delta_{E,g,b} = \Delta_{E' \otimes \Theta_{M,g,b'}} \star_b;$$

but also that  $w \in \Omega^q(M; E)_{\mathcal{B}}$  if and only if  $\star_b w \in \Omega^{m-q}(M; E' \otimes \Theta_M)_{\mathcal{B}'}$ , where  $\mathcal{B}'$  indicates the same operator in (4), see also (5), associated with  $E' \otimes \Theta_M$  and  $b'$  on the bordism  $(M, \partial_+ M, \partial_- M)'$ . In other words, the operator  $\star_b$  naturally *intertwines* the boundary value problems  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E, g, b}$  and  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)'}^{E' \otimes \Theta_M, g, b'}$ ; in particular, the roles of  $\partial_+ M$  and  $\partial_- M$  in each boundary value problem.

If  $\partial_+ M = \partial M$  and  $\partial_- M = \emptyset$  (resp.  $\partial_+ M = \emptyset$  and  $\partial_- M = \partial M$ ), then  $[\Delta, \mathcal{B}]_{(M, \partial M, \emptyset)}^{E, g, b}$  (resp.  $[\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)}^{E, g, b}$ ) is the boundary value problem where absolute (resp. relative) boundary conditions *only* are imposed on  $\partial M$ .

**1.4. Hermitian boundary value problems.** The operator  $\Delta_{E, g, b}$  was first considered on closed manifolds, by Burghelea and Haller in [BH07] to define a complex-valued version of the analytic Ray–Singer torsion. The analytic torsion, originally defined by Ray and Singer in [RS], needs instead a Hermitian Laplacian, obtained in an analog way to the one for the bilinear Laplacian. Indeed, by using a Hermitian structure  $h$  on  $E$ , instead of the bilinear form  $b$  all over in the considerations above, a Hermitian product on  $\Omega(M; E)$  is specified by  $\ll v, w \gg_{g, h} := \int_M \text{Tr}(v \wedge \star_h w)$ , the isomorphism  $\star_h$  being in this case induced by  $h$  and  $\star_g$ ; with respect to this inner product and after imposing absolute and relative boundary conditions a formally adjoint Laplacian  $\Delta_{E, g, h}$  is obtained, associated to the data  $\nabla^E, g$  and  $h$ , over the bordism  $(M, \partial_+ M, \partial_- M)$ . The space of smooth forms satisfying absolute/relative boundary conditions is denoted by  $\Omega(M; E)|_{\mathcal{B}}^h$ . In order to distinguish this problem from the bilinear one, we refer to it as the *Hermitian* boundary value problem. The Hermitian Laplacian acting on  $\Omega(M; E)|_{\mathcal{B}}^h$ , specifies an elliptic boundary value problem, denoted by  $(\Delta_{E, g, h}, \mathcal{B}_{E, g, h})$ ; this permits to extend the operator  $\Delta_{E, g, h} : \Omega(M; E)|_{\mathcal{B}}^h \subset L^2(M; E) \rightarrow \Omega(M; E) \subset L^2(M; E)$  the  $L^2$ -norm to a selfadjoint operator with domain of definition  $H_2(\Omega(M; E)|_{\mathcal{B}}^h)$ , that is, the  $H_2$ -Sobolev closure of  $\Omega(M; E)|_{\mathcal{B}}^h$ . For these facts, see [Lü], [Mü78], [Gi84] and [Gi04]. There exist well-known Hodge-decomposition results. Let us denote by  $\mathcal{H}_{\Delta_{\mathcal{B}}}^q(M; E)$  the space  $\ker(\Delta_{E, g, h}) \cap \Omega^q(M; E)|_{\mathcal{B}}^h$  of  $q$ -*Harmonic forms* satisfying boundary conditions. Then, by using the notation from (4) to (6) in the Hermitian setting, we have

$$(7) \quad \begin{aligned} \Omega^q(M; E)|_{\mathcal{B}^0}^h &= \mathcal{H}_{\Delta_{\mathcal{B}}}^q(M; E) \oplus d_E(\Omega^{q-1}(M; E)|_{\mathcal{B}^0}^h) \oplus d_{E, g, h}^*(\Omega^{q+1}(M; E)|_{\mathcal{B}^0}^h) \\ \mathcal{H}_{\Delta_{\mathcal{B}}}^q(M; E) &\cong H^q(M, \partial_- M; E), \end{aligned}$$

see Theorem 1.10 in [Lü] and page 239 in [Mü78]. This in turn permits to define the Ray–Singer metric on manifolds with boundary under absolute and relative boundary conditions. The Hermitian Laplacian has been studied by many authors, see for instance [RS], [Lü], [Mü78], [DF]. Moreover, in [BM06] Brüning and Ma studied the case  $\partial_- M = \emptyset$  and they obtained corresponding

anomaly formulas for the Ray–Singer metric. More recently, in [BM11], Brüning and Ma continued the study of the Hermitian Laplacian when  $\partial_{\pm}M$  are not necessarily empty.

**1.5. The spectrum of the bilinear Laplacian.** To study ellipticity for boundary value problems we use *ellipticity with respect to a conic subset of  $\mathbb{C}$* . The boundary valued problem  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E, g, b}$  is elliptic with respect to the cone  $\mathbb{C} \setminus (0, \infty)$ , see [Gi04]. Boundary ellipticity guarantees the existence of elliptic estimates, see Theorem 6.3.1 in [A] and Theorem 20.1.2 in [Hö]. These in turn permit to conclude that the unbounded operator

$$(8) \quad \Delta_{\mathcal{B}} : \mathcal{D}(\Delta_{\mathcal{B}}) \subset L^2(M; E) \rightarrow L^2(M; E)$$

with domain  $\mathcal{D}(\Delta_{\mathcal{B}}) := H_2(\Omega(M; E)|_{\mathcal{B}})$ , is closed in the  $L^2$ -norm and coincides with the  $L^2$ -closure extension of

$$\Delta_{E, g, b} : \Omega(M; E)|_{\mathcal{B}} \subset L^2(M; E) \rightarrow \Omega(M; E) \subset L^2(M; E),$$

regarded as unbounded operator on  $L^2(M; E)$ . Boundary ellipticity with respect to the cone  $\mathbb{C} \setminus (0, \infty)$  implies the following result, which can be found in a much more general setting in [Gb], specifically, see Remark 4 and then Theorem 3.3.2 and Corollary 3.3.3 in [Gb] (see also Remark 3.3.4 and discussion in section 1.5 in [Gb]).

**Lemma 2.** *Let  $\Delta_{\mathcal{B}}$  be the unbounded operator with domain of definition  $\mathcal{D}(\Delta_{\mathcal{B}})$  given in (8). This operator is densely defined in  $L^2(M; E)$ , possesses a non-empty resolvent set, its resolvent is compact and its spectrum is discrete. More precisely, for every  $\theta > 0$ , there exists  $R > 0$  such that  $\mathbb{B}_R(0)$ , the closed ball in  $\mathbb{C}$  centered at 0 and radius  $R$ , contains at most a finite subset of  $\text{Spec}(\Delta_{\mathcal{B}})$ ; the remaining part of the spectrum is entirely contained in the sector*

$$\Lambda_{R, \theta} := \{z \in \mathbb{C} \mid -\theta < \arg(z) < \theta \text{ and } |z| \geq R\}.$$

*Furthermore, for every  $\lambda \notin \Lambda_{R, \theta}$  large enough, there is  $C > 0$ , for which  $\|(\Delta_{\mathcal{B}} - \lambda)^{-1}\|_{L^2} \leq C/|\lambda|$ .*

**1.6. Generalized eigenspaces.** Since  $\text{Spec}(\Delta_{\mathcal{B}})$  is discrete, for each  $\lambda \in \text{Spec}(\Delta_{\mathcal{B}})$ , we choose  $\gamma(\lambda)$  a closed counter-clock-wise oriented curve surrounding  $\lambda$  as the unique point of  $\text{Spec}(\Delta_{\mathcal{B}})$ ; then consider the Riesz or spectral projection corresponding to  $\lambda$ :

$$(9) \quad \begin{aligned} P_{\Delta_{\mathcal{B}}}(\lambda) : L^2(M; E) &\rightarrow \mathcal{D}(\Delta_{\mathcal{B}}) \subset L^2(M; E), \\ w &\mapsto -(2\pi i)^{-1} \int_{\gamma(\lambda)} (\Delta_{\mathcal{B}} - \mu)^{-1} w d\mu, \end{aligned}$$

where the integral above converges uniformly in the  $L^2$ -norm as the limit of Riemann sums, since the function  $x \mapsto (\Delta_{\mathcal{B}} - x)^{-1}$  is analytic in a neighborhood of  $\gamma(\lambda)$ . The image of  $P_{\Delta_{\mathcal{B}}}(\lambda)$  in  $L^2(M; E)$  is denoted by



$\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) := P_{\Delta_{\mathcal{B}}}(\lambda)(L^2(M; E))$ . Since the resolvent of  $\Delta_{\mathcal{B}}$  is compact, the operator  $P_{\Delta_{\mathcal{B}}}(\lambda)$  is bounded on  $L^2(M; E)$ , and the space  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  is of finite dimension, see Theorem 6.29 in Chapter III, section 6 in [K]. Let us denote by  $(I - P_{\Delta_{\mathcal{B}}}(\lambda)) : L^2(M; E) \rightarrow L^2(M; E)$  the complementary projection to  $P_{\Delta_{\mathcal{B}}}(\lambda)$  on  $L^2(M; E)$ , and its image by  $\text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda)) := (I - P_{\Delta_{\mathcal{B}}}(\lambda))(L^2(M; E))$ . Then the space  $L^2(M; E)$  decomposes as a direct sum of Hilbert spaces compatible with the projections  $P_{\Delta_{\mathcal{B}}}(\lambda)$  and  $(I - P_{\Delta_{\mathcal{B}}}(\lambda))$ . More precisely, the following Lemma is a direct application of Theorem 6.17, page 178 in [K].

**Lemma 3.** *The unbounded operator  $\Delta_{\mathcal{B}}$  commute with  $P_{\Delta_{\mathcal{B}}}(\lambda)$ ; that is, for  $u \in \mathcal{D}(\Delta_{\mathcal{B}})$ , we have*

$$P_{\Delta_{\mathcal{B}}}(\lambda)u \in \mathcal{D}(\Delta_{\mathcal{B}}) \text{ and } P_{\Delta_{\mathcal{B}}}(\lambda)\Delta_{\mathcal{B}}u = \Delta_{\mathcal{B}}P_{\Delta_{\mathcal{B}}}(\lambda)u.$$

*Then,  $L^2(M; E)$  decomposes as*

$$L^2(M; E) \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \oplus \text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda)),$$

*such that  $P_{\Delta_{\mathcal{B}}}(\lambda)(\mathcal{D}(\Delta_{\mathcal{B}})) \subset \mathcal{D}(\Delta_{\mathcal{B}})$ ,  $\Delta_{\mathcal{B}}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)) \subset \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  and  $\Delta_{\mathcal{B}}(\text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda))) \subset \text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda))$ . Finally, the operator*

$$(10) \quad \Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)} : \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \rightarrow \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda),$$

*is bounded on  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$ ,  $\text{Spec}(\Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)}) = \{\lambda\}$  and the operator*

$$(11) \quad \Delta_{\mathcal{B}}|_{\text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda))} : \mathcal{D}((\Delta_{\mathcal{B}} - \lambda)|_{\text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda))}) \subset L^2(M; E) \rightarrow \text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda)),$$

*with  $\mathcal{D}((\Delta_{\mathcal{B}} - \lambda)|_{\text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda))}) = \text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda)) \cap \mathcal{D}(\Delta_{\mathcal{B}})$ , is invertible; in other words, the spectrum of  $\Delta_{\mathcal{B}}|_{\text{Im}(I - P_{\Delta_{\mathcal{B}}}(\lambda))}$  is exactly  $\text{Spec}(\Delta_{\mathcal{B}}) \setminus \{\lambda\}$ .*

The operator  $\Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)}$  in (10) is bounded and its spectrum contains 0 only; this together with  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  being of finite dimension, implies that  $\Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)}$  is nilpotent. Moreover, since the operators  $P_{\Delta_{\mathcal{B}}}(\lambda)$  and  $\Delta_{\mathcal{B}}$  commute and  $\mathcal{D}(\Delta_{\mathcal{B}})$  is invariant under  $\Delta_{\mathcal{B}}$ , the existence of elliptic estimates, Sobolev embedding and a standard argument of induction imply that if  $w \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \subset \mathcal{D}(\Delta_{\mathcal{B}}) \subset H_2(\Omega(M; E))$ , then  $w \in \Omega(M; E)|_{\mathcal{B}} \subset \Omega(M; E)$ . Thus each  $\lambda$ -eigenspace can be described as

$$\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) = \left\{ w \in \Omega(M; E)|_{\mathcal{B}} \left| \begin{array}{l} (\Delta_{E,g,b} - \lambda)^n w \in \Omega(M; E)|_{\mathcal{B}}, \forall n \geq 0, \\ \exists N \in \mathbb{N} \text{ s.t. } (\Delta_{E,g,b} - \lambda)^n w = 0, \forall n \geq N \end{array} \right. \right\}.$$

**Lemma 4.** *The space  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  is invariant under  $d_E$  and  $d_{E,g,b}^\sharp$ .*

*Proof.* We show that  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  is invariant under  $d_E$  and  $d_{E,g,b}^\sharp$ . Since  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  contains smooth differential forms only, it suffices to show that  $d_E w$  satisfies the boundary condition, whenever  $w \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$ . On  $\partial_+ M$ , the absolute part of the boundary, this immediately follows from

$d_E^2 = 0$ . Let us turn to  $\partial_- M$ , the relative part of the boundary. But, we know that the Riesz projections are well defined as bounded operators and they commute with the Laplacian on its domain of definition. That is,  $\Delta_{E,g,b}w$  lies in  $\Omega_{\Delta_B}(M; E)(\lambda)$  as well; in particular, it satisfies relative boundary conditions on  $\partial_- M$ , so that  $i_-^*(\Delta_{E,g,b}w) = 0$ . Together with  $i_-^*d_{E,g,b}^\sharp w = 0$ , this implies  $i_-^*d_{E,g,b}^\sharp d_E w = 0$ , hence  $d_E w$  also satisfies relative boundary conditions. Finally, the corresponding statement for  $d_{E,g,b}^\sharp$  follows by the duality between the absolute and relative boundary operators.  $\square$

Consider the space

$$\Omega_{\Delta_B}(M; E)(\lambda)^c := \Omega(M; E) \cap \text{Im}(I - P_{\Delta_B}(\lambda)).$$

Invertibility of the operator in (11) together with the existence of elliptic estimates imply that the restriction of  $(\Delta_B - \lambda)$  to the space of forms in  $\Omega_{\Delta_B}(M; E)(\lambda)^c$ , satisfying boundary conditions (see notation in (4)), provides an isomorphism

$$(12) \quad (\Delta_B - \lambda)|_{\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}}} : \Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}} \rightarrow \Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}}.$$

### 1.7. Orthogonality and Hodge decomposition for smooth forms.

**Lemma 5.** *For  $\lambda \in \text{Spec}(\Delta_B)$  and  $v, w \in L^2(M; E)$ , we have the formula  $\beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}(v, P_{\Delta_B}(\lambda)w)$ .*

*Proof.* Since  $\beta_{g,b}$  continuously extends to a nondegenerate bilinear form on  $L^2(M; E)$ , it is enough to prove the statement on smooth forms. For  $v, w \in \Omega(M; E)$ , we have

$$-2\pi i \beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}\left(\int_{\gamma_\lambda} (\Delta_B - \mu)^{-1} v d\mu, w\right) = \int_{\gamma_\lambda} \beta_{g,b}\left((\Delta_B - \mu)^{-1} v, w\right) d\mu,$$

where the last equality above holds, as  $\int_{\gamma_\lambda}$  converges uniformly in the  $L^2$ -norm. Since  $\gamma_\lambda \cap \text{Spec}(\Delta_B) = \emptyset$ , we have  $(\Delta_B - \mu)^{-1}w \in \mathcal{D}(\Delta_B)$  so that  $w = (\Delta_B - \mu)(\Delta_B - \mu)^{-1}w$  for each  $\mu \in \gamma_\lambda$ . Now, from the isomorphism in (12), both  $(\Delta_B - \mu)^{-1}v$  and  $(\Delta_B - \mu)^{-1}w$  belong in fact to  $\Omega_{\Delta_B}(M; E)(\lambda)^c|_{\mathcal{B}}$ , so we can apply Lemma 1 and obtain

$$\begin{aligned} \beta_{g,b}\left((\Delta_B - \mu)^{-1}v, w\right) &= \beta_{g,b}\left((\Delta_B - \mu)^{-1}v, (\Delta_{E,g,b} - \mu)(\Delta_B - \mu)^{-1}w\right) \\ &= \beta_{g,b}\left((\Delta_{E,g,b} - \mu)(\Delta_B - \mu)^{-1}v, (\Delta_B - \mu)^{-1}w\right) \\ &= \beta_{g,b}\left(v, (\Delta_B - \mu)^{-1}w\right); \end{aligned}$$

that is,  $\beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = -(-2\pi i)^{-1} \int_{\gamma_\lambda} \beta_{g,b}(v, (\Delta_B - \mu)^{-1}w) d\mu$  and hence the equality  $\beta_{g,b}(P_{\Delta_B}(\lambda)v, w) = \beta_{g,b}(v, P_{\Delta_B}(\lambda)w)$  holds.  $\square$

**Proposition 1.** *We have a  $\beta$ -orthogonally direct sum decomposition:*

$$(13) \quad \Omega(M; E) \cong \Omega_{\Delta_B}(M; E)(\lambda) \oplus \Omega_{\Delta_B}(M; E)(\lambda)^c,$$

If  $\lambda, \mu \in \text{Spec}(\Delta_{\mathcal{B}})$  with  $\lambda \neq \mu$ , then  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\mu) \perp_{\beta} \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$ . In particular,  $\beta$  restricts to each of these subspaces as a non degenerate symmetric bilinear form. Furthermore, with the notation in section 1.2, there is a  $\beta$ -orthogonal direct decomposition

$$(14) \quad \Omega(M; E)|_{\mathcal{B}^0} \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \oplus \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0},$$

which is invariant under  $d_E$ .

*Proof.* Remark that  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) = P_{\Delta_{\mathcal{B}}}(\lambda)(\Omega(M; E))$ , and therefore the decomposition follows from the direct sum decomposition of  $L^2(M; E)$ , see Lemma 3. We show that  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \perp_{\beta} \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c$ : take  $v \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  and  $w \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c$ , so that

$$\beta_{g,b}(v, w) = \beta_{g,b}(P_{\Delta_{\mathcal{B}}}(\lambda)v, w) = \beta_{g,b}(v, P_{\Delta_{\mathcal{B}}}(\lambda)w) = 0,$$

where the second equality follows from Lemma 5 and the last one is true because  $w$  is in the image of the complementary projection of  $P_{\Delta_{\mathcal{B}}}(\lambda)$ . Since  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \subset \Omega(M; E)|_{\mathcal{B}^0}$ , the decomposition in (13) implies directness and  $\beta$ -orthogonality for that in (14). By Lemma 4,  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  is invariant under  $d_E$  (and  $d_{E,g,b}^{\sharp}$ ). But,  $d_E(\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0}) \subset \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0}$  as well as it can be checked by using the Green's formulas from Lemma 3, that  $d_{E,g,b}^{\sharp}$  leaves invariant  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  and  $\beta$ -orthogonality of (13).  $\square$

**Corollary 1.** *For  $\lambda \in \text{Spec}(\Delta_{\mathcal{B}})$  and with the notation in (6), consider the space  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0}$ . Then, the spaces  $d_E(\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0})$  and  $d_{E,g,b}^{\sharp}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0})$  are  $\beta$ -orthogonal to  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$ .*

*Proof.* If  $u \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  and  $v \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)^c|_{\mathcal{B}^0}$ , then, by using Lemma 1, invariance of  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)$  under  $d_{E,g,b}^{\sharp}$  (see also Lemma 4 and Proposition 1 above), we have  $\beta_{g,b}(u, d_E v) = \beta_{g,b}(d_{E,g,b}^{\sharp} u, v) = 0$ . The proof for  $d_{E,g,b}^{\sharp}$  is analog.  $\square$

**Corollary 2.** *(Hodge decomposition) We have the  $\beta_{g,b}$ -orthogonal decomposition  $\Omega(M; E) \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \oplus \Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}})$ .*

*Proof.* This follows from Proposition 1 and the isomorphism in (12).  $\square$

The following result is analog to Proposition 2.1 in [BFK];

**Proposition 2.** *The following are  $\beta_{g,b}$ -orthogonal direct sum decompositions:*

- (i)  $\Omega(M; E) \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \oplus d_E(d_{E,g,b}^{\sharp}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}})) \oplus d_{E,g,b}^{\sharp}(d_E(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}}))$ ,
- (ii)  $\Omega(M; E)|_{\mathcal{B}^0} \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \oplus d_E(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}^0}) \oplus d_{E,g,b}^{\sharp}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}})$ ,
- (iii)  $\Omega(M; E)|_{\mathcal{B}^0} \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \oplus d_E(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}}) \oplus d_{E,g,b}^{\sharp}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c|_{\mathcal{B}})$ .

Moreover, the restriction of  $\beta_{g,b}$  to each of the spaces appearing above is nondegenerate.

*Proof.* We prove (i). From Corollary 2, every  $u \in \Omega(M; E)$  can be written as  $u = u_0 + d_E(d_{E,g,b}^\sharp u) + d_{E,g,b}^\sharp(d_E u)$ , with  $u_0 \in \Omega_{\Delta_B}(M; E)(0)$  and  $u \in \Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}}$ . That

$$d_E(d_{E,g,b}^\sharp(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}})) \perp_{\beta_{g,b}} d_{E,g,b}^\sharp(d_E(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}})),$$

follows from Lemma 1 and  $d_E^2 = 0$ . To see that (i) is a direct sum, we check that the intersection of the last two spaces on the right of (i) is trivial. So, take  $u \in \Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}}$ , and suppose there are  $v, w \in \Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}}$  with  $u = d_E(d_{E,g,b}^\sharp v) = d_{E,g,b}^\sharp(d_E w)$ . Remark obviously that  $\Delta_{E,g,b} u = 0$  but also that  $u \in \Omega_{\Delta_B}(M; E)(0)$ , since

- (a)  $i_-^* u = d_E(i_-^* d_{E,g,b}^\sharp v) = 0$ , as  $v$  satisfies boundary conditions,
- (b)  $i_-^* d_{E,g,b}^\sharp u = i_-^* d_{E,g,b}^\sharp d_{E,g,b}^\sharp d_E v = 0$ ,
- (c)  $i_+^* \star_b u = \pm d_E(i_+^* d_{E,g,b}^\sharp \star_b w) = 0$ ; as  $w$  satisfies boundary conditions,
- (d)  $i_+^* d_{E,g,b}^\sharp \star_b u = \pm i_+^* \star_b d_E(d_{E,g,b}^\sharp v) = 0$ ;

therefore, from Proposition 1,  $u$  must vanish, so that the sum in (i) is direct. This decomposition is clearly  $\beta$ -orthogonal. The decompositions in (ii) and (iii) follow from that in (i), Lemma 1, the isomorphism in (12) and the definition of boundary conditions as we have proceeded to prove the statement (i); we omit the details. Since  $d_E(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}}) \subset d_E(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}})$ , directness of decomposition (iii) follows from that of (ii). To check directness in (ii), firstly observe that by Proposition 1 we have  $d_E(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}^0_-}) \subset \Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}^0_-}$  and therefore the space  $\Omega_{\Delta_B}(M; E)(0) \cap d_E(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}^0_-})$  is trivial. Secondly, from the inclusion  $\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}} \subset \Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}^0}$ , Corollary 1 and Proposition 1, the intersection of  $\Omega_{\Delta_B}(M; E)(0)$  and  $d_{E,g,b}^\sharp(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}})$  is also trivial. Thirdly, the intersection of  $d_E(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}^0_-})$  and the space  $d_{E,g,b}^\sharp(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}})$  is trivial as well; indeed if  $u \in \Omega_{\Delta_B}(M; E)(0)^c$  is such that  $u = d_E v$  for  $v \in \Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}^0_-}$  and  $u = d_{E,g,b}^\sharp w$  for  $w \in d_{E,g,b}^\sharp(\Omega_{\Delta_B}(M; E)(0)^c|_{\mathcal{B}})$ , then, it follows that  $u \in \Omega_{\Delta_B}(M; E)(0)$ , and therefore  $u = 0$ . From Lemma 1, the decompositions in (ii) and (iii) are  $\beta_{g,b}$ -orthogonal. This, together with directness of these decompositions, implies that  $\beta_{g,b}$  restricts to each space appearing on the right hand side of (ii) and (iii) as a nondegenerate bilinear form. Finally, since  $\beta_{g,b}$  is nondegenerate on  $\Omega_0(M; E)$ , the space of smooth forms compactly supported in the interior of  $M$ , it can be continuously extended to a nondegenerate bilinear form on  $\Omega(M; E)|_{\mathcal{B}^0_-}$  and  $\Omega(M; E)|_{\mathcal{B}^0}$  respectively.

□

**1.8. Cohomology.** With the notation in section 1.2, recall that the cochain complex  $(\Omega(M; E)|_{\mathcal{B}^0}, d_E)$  computes *De-Rham cohomology* of  $M$  relative to  $\partial_- M$ , with coefficients on  $E$ , see for instance [BT]. For  $\lambda \in \text{Spec}(\Delta_{\mathcal{B}})$ ,  $(\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda), d_E)$  is cochain subcomplex; consider the inclusion

$$\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda) \hookrightarrow \Omega(M; E)|_{\mathcal{B}^0}.$$

From Lemma 3, Lemma 4 and the isomorphism in (12), every generalized eigenspace corresponding to a *non-zero* eigenvalue is acyclic, that is,  $H(\Omega_{\Delta_{\mathcal{B}}}(M; E)(\lambda)) = 0$ . Furthermore, for  $\lambda = 0$ , we have the following.

**Proposition 3.** *The inclusion  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \hookrightarrow \Omega(M; E)|_{\mathcal{B}^0}$  induces an isomorphism in cohomology:  $H^*(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)) \cong H^*(M, \partial_- M, E)$ .*

*Proof.* Since  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \subset \Omega(M; E)|_{\mathcal{B}^0}$ , the space  $\Omega(M; E)|_{\mathcal{B}^0}$  admits a decomposition compatible with the one in Corollary 2 and therefore it decomposes as

$$\Omega(M; E)|_{\mathcal{B}^0} \cong \Omega_{\Delta_{\mathcal{B}}}(M; E)(0) \oplus \Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}}) |_{\mathcal{B}^0},$$

where  $\Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}}) |_{\mathcal{B}^0}$  is also a cochain subcomplex, because of Proposition 1 and that  $\Omega(M; E)|_{\mathcal{B}^0}$  is invariant under the action of  $d_E$ . Thus the assertion is true, if the corresponding cohomology groups vanish; that is, if every closed form  $w$  in  $\Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}}) |_{\mathcal{B}^0}$  is also exact. By Proposition 2.(ii), there are  $w_1 \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}^0}$  and  $w_2 \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}}$  such that  $w = d_E w_1 + d_{E,g,b}^\sharp w_2$ . First, we claim that  $\beta_{g,b}(d_{E,g,b}^\sharp w_2, v_1) = 0$ , for all  $v_1 \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}^0}$ , see (6); indeed, from Proposition 2.(i), there exist  $v_2, u_2 \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}}$ , such that  $v_1 = d_E v_2 + d_{E,g,b}^\sharp u_2$  and hence  $\beta_{g,b}(d_{E,g,b}^\sharp w_2, d_E v_2 + d_{E,g,b}^\sharp u_2) = 0$ , where we have used that  $d_{E,g,b}^\sharp w_2, d_E v_2$  and  $d_{E,g,b}^\sharp u_2 \in \Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}^0}$ , Lemma 1,  $(d_{E,g,b}^\sharp)^2 = 0$  and that  $\beta_{g,b}(d_E d_{E,g,b}^\sharp w_2, u_2)$  vanishes, because  $w$  being close implies  $d_E d_{E,g,b}^\sharp w_2 = 0$ . Finally, since  $d_{E,g,b}^\sharp w_2$  belongs to  $\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}^0}$  as well, and that  $\beta_{g,b}$  restricted to this sub-space is also nondegenerate, see Proposition 2, from the claim above, we have  $d_{E,g,b}^\sharp w_2 = 0$ . That is,  $w$  is exact in  $\Delta_{E,g,b}(\Omega_{\Delta_{\mathcal{B}}}(M; E)(0)^c |_{\mathcal{B}}) |_{\mathcal{B}^0}$ .

□

## 2. HEAT TRACE ASYMPTOTIC EXPANSION AND ANOMALY FORMULAS

### 2.1. Heat trace asymptotic coefficients as polynomial invariants.

Consider more generally a boundary value problem  $(D, \mathcal{B})$ , where  $D$  is an operator of Laplace type,  $\mathcal{B}$  a boundary operator specifying *mixed boundary*

conditions and denote by  $D_{\mathcal{B}}$  its  $L^2$ -realization. Then, for  $t > 0$ , the heat kernel  $\exp(-tD_{\mathcal{B}})$  is a smoothing operator and of trace class in  $L^2$ -norm; moreover, for  $\psi \in \Gamma(M; \text{End}(\Lambda^*(T^*M) \otimes E))$  and  $t \rightarrow 0$  we have a complete asymptotic expansion:  $\text{Tr}_{L^2}(\psi e^{-tD_{\mathcal{B}}}) \sim \sum_{n=0}^{\infty} a_n(\psi, D, \mathcal{B}) t^{(n-m)/2}$ , where  $a_n(\psi, D, \mathcal{B})$  are the *heat trace asymptotic coefficients* and they are computed by the formula

$$(15) \quad a_n(\psi, D, \mathcal{B}) = \int_M \text{Tr}(\psi \cdot \mathfrak{e}_n(D)) \text{vol}_g(M) + \sum_{k=0}^{n-1} \int_{\partial M} \text{Tr}(\nabla_{\varsigma_{\text{in}}}^k \psi \cdot \mathfrak{e}_{n,k}(D, \mathcal{B})) \text{vol}_g(\partial M)$$

where  $\nabla_{\varsigma_{\text{in}}}^k$  denotes the  $k$ -covariant derivative along the inwards pointing geodesic unit vector field normal to  $\partial M$ , computed with respect to the Levi-Civita connection and an auxiliary one on  $E$ ; the quantities  $\mathfrak{e}_n(x, D)$  and  $\mathfrak{e}_{n,k}(y, D, \mathcal{B})$  are locally computable invariant endomorphism-valued forms; for all these facts, see Theorem 1.4.5. in [Gi04] (see also Chapter 3 in [Gi04], [Se69] and [Se67]). We are interested in the constant coefficient in the heat asymptotic expansion, denoted by

$$\text{LIM}_{t \rightarrow 0}(\text{Tr}_{L^2}(\psi e^{-tD_{\mathcal{B}}})) := a_m(\psi, D, \mathcal{B}),$$

see [BGV]. To compute these coefficients, we have to integrate traces of invariant endomorphism-valued forms over the interior of  $M$  and its boundary. These forms are locally computable as polynomials in the jets of the symbol of  $D$  and  $\mathcal{B}$ , which in turn can be computed as universal polynomial invariants in locally computable tensorial objects. This is achieved by using Weyl theory of invariants and discussed in greater detail in the work of Gilkey, see sections 1.7-1.8 in [Gi04] (see also sections 1.7, 1.9 and 4.8 in [Gi84]).

The Laplace type operator  $D$  is invariantly characterized by a unique connection  $\nabla^D$  and a bundle endomorphism  $E^D \in \Gamma(M; \text{End}(\Lambda^*T^*M \otimes E))$ , see Lemma 1.2.1 in [Gi04]. The operator  $\mathcal{B}$  imposing mixed boundary conditions can be invariantly described by an *involution*  $\chi^{\mathcal{B}} \in \Gamma(\partial M; \text{End}(i^*(\Lambda^*(T^*M) \otimes E)))$ , extended parallelly along the normal direction to a collared neighborhood of  $\partial M$  in  $M$ , and a bundle endomorphism  $S^{\mathcal{B}} \in \Gamma(M; \text{End}(V_+))$ , where  $V_+ := 1/2(1 + \chi^{\mathcal{B}})(\Lambda^*(T^*M) \otimes E)$ ; for much more details about these objects see section 1.2.2 and sections 1.5.3–1.5.5 in [Gi04]. The following summarizes Lemmas 3.1.10 and 3.1.11 in section 3.1.8 in [Gi04]; see also sections 3.6, 1.7, 1.9 and 4.8 in [Gi84].

**Lemma 6.** *Let  $\nabla^D, E^D, \chi^{\mathcal{B}}$  and  $S^{\mathcal{B}}$  as above. With respect to an orthonormal frame on  $TM$ , we denote by  $;$  the components of multiple covariant differentiation of tensor fields of all types with respect to the connection  $\nabla^D$  and the Levi-Civita connection  $\nabla$ . Let  $R_{i_1 i_2 i_3 i_4}$  be the components of the curvature tensor corresponding to  $\nabla$ ,  $\Omega_{i_1 i_2}^D$  be those of the curvature tensor corresponding to  $\nabla^D$  and  $L_{a_1 a_2}$  be those of the second fundamental form (the indices  $i_1 i_2 i_3$  and  $i_4$  run from 1 to  $m$ , whereas  $a_1$  and  $a_2$  run from 1 to  $m-1$ ).*

Consider the endomorphism-valued forms  $\mathfrak{e}_n(D_u)$  and  $\mathfrak{e}_{n,k}(D_u, \mathcal{B}_u)$  in (15). Then

- (1) The quantities  $\mathfrak{e}_n(D)$  are polynomials belonging to the universal graded non commutative algebra generated by (germs of)  $R_{i_1 i_2 i_3 i_4; \dots}, \Omega_{i_1 i_2; \dots}^D, E^D, \dots$  and  $\text{id}$ .
- (2) The quantities  $\mathfrak{e}_{n,k}(D, \mathcal{B})$  are polynomials belonging to the universal graded non commutative algebra generated by (germs of)  $R_{i_1 i_2 i_3 i_4; \dots}, \Omega_{i_1 i_2; \dots}^D, E^D, \dots, L_{a_1 a_2; \dots}, \chi_{; \dots}^{\mathcal{B}}, S_{; \dots}^{\mathcal{B}}$  and  $\text{id}$ .

**2.2. Heat trace asymptotics for *Hermitian* boundary value problems.** Brüning and Ma studied the Hermitian Laplacian on a manifold with boundary, on which absolute boundary conditions were imposed, see [BM06]. They obtained anomaly formulas for the associated Ray–Singer analytic torsion seen as a metric on  $\det H(M; E)$ . Their formulas express the variation of the Ray–Singer metric with respect to smooth variations of the metric on  $M$  and of the Hermitian structure on  $E$ . This variation is encoded in the constant term in the heat kernel asymptotic expansion associated to the Hermitian boundary value problem under absolute boundary conditions. Recall the remarks and the notation from section 1.3.

First we recall some definitions and notation from [BM06]. We denote by  $\mathfrak{e}(M, g)$  the *Euler form* associated to the metric  $g$ ,  $\mathfrak{e}_{\mathbf{b}}(\partial M, g)$  and  $B(\partial M, g)$  the *characteristic forms on the boundary* defined by the formulas (1.17), page 775 in [BM06]. Given  $\{g_s\}_s$  a smooth path of Riemannian metrics on  $M$  connecting  $g_0$  and  $g_1$ , consider the *secondary forms*  $\tilde{\mathfrak{e}}(M, g_0, g_1)$  and  $\tilde{\mathfrak{e}}_{\mathbf{b}}(\partial M, g_0, g_1)$  of *Chern–Simons* type defined in (1.45), page 780 in [BM06].

**Proposition 4. (*Brüning–Ma*)** *Let  $(M, \partial M, \emptyset)$  be a compact Riemannian bordism. Consider  $[\Delta, \mathcal{B}]_{(M, \partial M, \emptyset)}^{E, g, h}$  the Hermitian boundary value problem and denote by  $\Delta_{\text{abs}, h}$  its  $L^2$ -realization. For  $\phi \in \Gamma(M, \text{End}(E))$  we have*

$$\text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \phi e^{-t \Delta_{\text{abs}, h}} \right) \right) = \int_M \text{Tr}(\phi) \mathfrak{e}(M, g) - (-1)^m \int_{\partial M} i^* \text{Tr}(\phi) \mathfrak{e}_{\mathbf{b}}(\partial M, g).$$

Moreover, let  $\xi \in \Gamma(M, \text{End}(TM))$  be a symmetric endomorphism with respect to the metric  $g$ , denote by  $\mathbf{D}^* \xi \in \Gamma(M, \text{End}(\Lambda^* T^* M))$  its extension as a derivation on  $\Lambda^*(T^* M)$  and set  $\Psi := \mathbf{D}^* \xi - \frac{1}{2} \text{Tr}(\xi)$ . For  $\tau \in \mathbb{R}$  taken small enough such that  $g + \tau g \xi$  is a nondegenerate symmetric metric on  $TM$ , we have

$$\begin{aligned} \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \Psi e^{-t \Delta_{\text{abs}, h}} \right) \right) &= -2 \int_M \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\mathfrak{e}}(M, g, g + \tau g \xi) \wedge \omega(\nabla^E, h) \\ &\quad + 2 \int_{\partial M} - \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\mathfrak{e}}_{\mathbf{b}}(\partial M, g, g + \tau g \xi) \wedge i^* \omega(\nabla^E, h) \\ &\quad + \text{rank}(E) \int_{\partial M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} B(\partial M, g + \tau g \xi), \end{aligned}$$

where  $\omega(\nabla^E, h) := -\frac{1}{2} \text{Tr}(h^{-1} \nabla^E h)$  is a real valued closed one-form.

*Proof.* First, since for any endomorphism  $\phi \in \Gamma(M, \text{End}(E))$ , there exist unique selfadjoint elements  $\phi_r, \phi_i \in \Gamma(M, \text{End}(E))$  such that  $\phi = \phi_r + \mathbf{i}\phi_i$ , it is enough to prove the first formula for a selfadjoint  $\phi$ . For  $\phi_u = h_u^{-1} \frac{\partial}{\partial u} h_u$  (resp.  $\Psi_u = g_u^{-1} \frac{\partial}{\partial u} g_u$ ) where  $h_u$  and (resp.  $g_u$ ) is a smooth one real parameter family of Hermitian and (resp. Riemannian) metrics with  $h_0 = h$  and (resp.  $g_0 = g$ ), the statement of the Proposition is the infinitesimal version of Brüning and Ma's results, see Theorem 4.6, expression (5.72), Section 5.5 and expressions (5.72) and (5.75) in [BM06]. However, these formulas still hold for arbitrary selfadjoint  $\phi_u$  (resp. arbitrary symmetric  $\Psi_u$ ). Indeed, remark that by taking small supports if necessary, the formulas above hold for the first order family  $\phi_u = \phi_0 + u \frac{\partial}{\partial s} \phi_s|_{s=0}$  for all  $u$  small enough.  $\square$

The following uses Poincaré duality to relate boundary value problems under absolute and relative boundary conditions.

**Lemma 7.** *Let  $\bar{E}'$  the dual of the conjugated complex vector bundle of  $E$ , endowed with dual flat connection and dual Hermitian form to those on  $E$ . Consider the compact Riemannian bordism  $(M, \emptyset, \partial M)$  together with its dual  $(M, \emptyset, \partial M)' := (M, \partial M, \emptyset)$ . We look at the Hermitian boundary value problem  $[\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)}^{E, g, h}$  with  $L^2$ -realization denoted by  $\Delta_{\text{rel}, h}$  and the its dual Hermitian boundary value problem  $[\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)'}^{\bar{E}' \otimes \Theta_{M, g, h'}}$  with corresponding  $L^2$ -realization  $\Delta'_{\text{abs}, h'}$ . If  $\phi, \xi$  and  $\Psi$  are as in Proposition 4, then*

$$(16) \quad \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \phi e^{-t \Delta_{\text{rel}, h}} \right) \right) = (-1)^m \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \phi^* e^{-t \Delta'_{\text{abs}, h'}} \right) \right),$$

where  $\phi^* := h \phi h^{-1}$ , and

$$(17) \quad \text{LIM}_{t \rightarrow 0} \text{STr} \left( \Psi e^{-t \Delta_{\text{rel}, h}} \right) = (-1)^{m+1} \text{LIM}_{t \rightarrow 0} \text{STr} \left( \Psi e^{-t \Delta'_{\text{abs}, h'}} \right).$$

*Proof.* We consider the complex vector bundle isomorphism between  $E$  and  $\bar{E}'$  provided by the Hermitian metric on  $E$  (see page 286 in [BT]), which we still denote by  $h \in \Omega^0(M; \text{End}(E, \bar{E}'))$ . With respect to the induced connection on  $\text{End}(E, \bar{E}')$ , we denote  $\nabla_X^E h \in \Omega^1(M; \text{End}(E, \bar{E}'))$ . By considering the Hermitian metric on  $E$  and the Riemannian metric on  $M$ , we obtain  $\star_h := \star \otimes h : \Omega(M; E) \rightarrow \Omega(M; \bar{E}' \otimes \Theta_M)$  a complex linear isomorphism used to define  $d_{E, g, h}^* := (-1)^q \star_h^{-1} d_{\bar{E}' \otimes \Theta_M} \star_h : \Omega^q(M; E) \rightarrow \Omega^{q-1}(M; E)$ ; this is the formal adjoint to  $d_E$  with respect to the Hermitian product on  $\Omega(M; E)$ . Moreover, the formula  $d_{\bar{E}'} d_{\bar{E}' \otimes \Theta_{M, g, h'}}^* \star_h = \star_h d_{E, g, h}^\sharp d_E$  holds and therefore  $\star_h \Delta_{E, g, h} := \Delta_{\bar{E}' \otimes \Theta_{M, g, h'}} \star_h$ . Moreover, as in Section 1.3, the operator  $\star_h$  intertwines  $E$ -valued forms satisfying relative (resp. absolute) boundary conditions with  $\bar{E}'$ -valued forms satisfying absolute (resp. relative) boundary



conditions. That is, the corresponding realizations verify

$$(18) \quad \Delta_{\text{rel},h} = \star_h^{-1} \Delta'_{\text{abs},h'} \star_h$$

and therefore  $\phi \exp(-t\Delta_{\text{rel},h}) = \star_h^{-1} \phi^* \exp(-t\Delta'_{\text{abs},h'}) \star_h$ , where  $\phi^* := h\phi h'$ . Thus, since the supertrace vanish on supercommutators of graded complex-linear operators and the degree of  $\star_{h,q}$  is  $m - q$ , we obtain for (16), the formula  $\mathbf{STr}(\phi e^{-t\Delta_{\text{rel},h}}) = (-1)^m \mathbf{STr}(\phi^* e^{-t\Delta'_{\text{abs},h'}})$ . We now prove (17):

$$(19) \quad \begin{aligned} \Psi(\star_q \otimes h)^{-1} &= ((\mathbf{D}^* \xi - \tfrac{1}{2} \text{Tr}(\xi)) \otimes 1) (\star_q \otimes h)^{-1} \\ &= (\star_q \otimes h)^{-1} ((\star_q (\mathbf{D}^* \xi - \tfrac{1}{2} \text{Tr}(\xi)) \star_q^{-1}) \otimes 1) \\ &= -(\star_q \otimes h)^{-1} ((\mathbf{D}^* \xi - \tfrac{1}{2} \text{Tr}(\xi)) \otimes 1) \\ &= -(\star_q \otimes h)^{-1} \Psi \end{aligned}$$

where we have used

$$(20) \quad \star_q (\mathbf{D}^* \xi - \tfrac{1}{2} \text{Tr}(\xi)) \star_q^{-1} = -\mathbf{D}^* \xi + \tfrac{1}{2} \text{Tr}(\xi).$$

To prove (20), it is enough to compute  $\star_q \mathbf{D}^* \xi \star_q^{-1}$ , since  $\text{Tr}(\xi)$  commutes with  $\star$ . We evaluate this operator on a generic element of a particular choice of an orthonormal local frame for  $\Lambda^* T^* M$ . We fix this frame as follows. Since  $\xi$  is a symmetric complex endomorphism of  $TM$ , we can choose an orthonormal local frame  $\{e_i\}_1^m$  such that  $\xi e_i = \lambda_i e_i$  and we locally fix an orientation for this frame so that  $\{e^{i_1} \wedge \dots \wedge e^{i_q}\}_{1 \leq i_1 < \dots < i_q \leq m}$  is positive oriented local frame for  $\Lambda^* T^* M$ . Under these assumptions,  $\star_q (e^{i_1} \wedge \dots \wedge e^{i_q}) = e^{j_1} \wedge \dots \wedge e^{j_{m-q}}$ , where the ordered indices  $(j_1, \dots, j_{m-q})$  is the unique possible complementary shuffle corresponding to the ordered choice of indices  $(i_1, \dots, i_q)$ . Therefore, we compute

$$\begin{aligned} \star_q \mathbf{D}^* \xi \star_q^{-1} (e^{j_1} \wedge \dots \wedge e^{j_{m-q}}) &= \star_q \mathbf{D}^* \xi (e^{i_1} \wedge \dots \wedge e^{i_q}) \\ &= \star_q \sum_{l=1}^q (e^{i_1} \wedge \dots \wedge \xi(e^{i_l}) \wedge \dots \wedge e^{i_q}) \\ &= \star_q \sum_{l=1}^q \lambda_{i_l} (e^{i_1} \wedge \dots \wedge e^{i_l} \wedge \dots \wedge e^{i_q}) \\ &= \sum_{l=1}^q \lambda_{i_l} (e^{j_1} \wedge \dots \wedge e^{j_{m-q}}) \\ &= \sum_{l=1}^m \lambda_{i_l} (e^{j_1} \wedge \dots \wedge e^{j_{m-q}}) - \sum_{l=1}^{m-q} \lambda_{j_l} (e^{j_1} \wedge \dots \wedge e^{j_{m-q}}) \\ &= \sum_{l=1}^m \lambda_{i_l} (e^{j_1} \wedge \dots \wedge e^{j_{m-q}}) - \sum_{l=1}^{m-q} (e^{j_1} \wedge \dots \wedge \lambda_{j_l} e^{j_l} \wedge \dots \wedge e^{j_{m-q}}) \\ &= (\text{Tr} \xi - \mathbf{D}^* \xi) (e^{j_1} \wedge \dots \wedge e^{j_{m-q}}) \end{aligned}$$

and we obtain (20). Finally, we use (19) to pass to the conjugated complex; hence with (18) and duality between these boundary value problems we obtain

$$\Psi \exp(-t\Delta_{\text{rel},h}) = \Psi \star_h^{-1} \exp(-t\Delta'_{\text{abs},h'}) \star_h = -\star_h^{-1} \Psi \exp(-t\Delta'_{\text{abs},h'}) \star_h$$

thus, as for (16), we have

$$\mathbf{STr}(\Psi \exp(-t\Delta_{\text{rel},h})) = -(-1)^m \mathbf{STr}(\Psi \exp(-t\Delta'_{\text{abs},h'}))$$

□

**Proposition 5.** *For the Riemannian bordism  $(M, \emptyset, \partial M)$ , consider the Hermitian boundary value problem  $[\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)}^{E, g, h}$  with its  $L^2$ -realization denoted by  $\Delta_{\text{rel}, h}$ . If  $\phi$ ,  $\xi$  and  $\Psi$  are as in Proposition 4, then*

$$\text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \phi e^{-t \Delta_{\text{rel}, h}} \right) \right) = \int_M \text{Tr}(\phi) \mathbf{e}(M, g) - \int_{\partial M} i^* \text{Tr}(\phi) \mathbf{e}_{\mathbf{b}}(\partial M, g).$$

and

$$\begin{aligned} \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \Psi e^{-t \Delta_{\text{rel}, h}} \right) \right) &= -2 \int_M \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\mathbf{e}}(M, g, g + \tau g \xi) \wedge \omega(\nabla^E, h) \\ &\quad + 2(-1)^{m+1} \int_{\partial M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g, g + \tau g \xi) \wedge i^* \omega(\nabla^E, h) \\ &\quad + (-1)^{m+1} \text{rank}(E) \int_{\partial M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} B(\partial M, g + \tau g \xi). \end{aligned}$$

*Proof.* Recall that  $w \in \Omega^*(M; E)$  satisfies relative boundary conditions if and only if the smooth form  $\star_h w \in \Omega^{m-*}(M; \bar{E}' \otimes \Theta_M)$  satisfies absolute boundary conditions on  $\partial M$ ; hence, the first formula follows from formula (16) in Lemma 7, and the results from Brüning and Ma for the Hermitian Laplacian stated in Proposition 4. The second formula follows from Lemma formula (17) in 7, Proposition 4 and  $\omega(\nabla^E, h) = -\omega(\nabla^{E'}, h')$ , see for instance section 2.4 in [BH07]. □

The following Lemma is a direct consequence of Lemma 6 and disjointness of  $\partial_+ M$  and  $\partial_- M$ .

**Lemma 8.** *For  $(M, \partial M, \emptyset)$ ,  $(M, \emptyset, \partial M)$  and  $(M, \partial_+ M, \partial_- M)$  let us consider  $[\Delta, \mathcal{B}]_{(M, \partial M, \emptyset)}^{E, g, h}$ ,  $[\Delta, \mathcal{B}]_{(M, \emptyset, \partial M)}^{E, g, h}$  and  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E, g, h}$  the corresponding Hermitian boundary value problems, together with their  $L^2$ -realizations  $\Delta_{\text{abs}, h}$ ,  $\Delta_{\text{rel}, h}$  and  $\Delta_{\mathcal{B}, h}$ , respectively. Let  $\psi_{\pm} \in \Gamma(M; \text{End}(\Lambda^*(T^*M) \otimes E))$  be chosen in such a way that  $\text{supp}(\psi_{\pm}) \cap \partial_{\mp} M = \emptyset$ , then*

$$\text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \psi_+ e^{-t \Delta_{\mathcal{B}, h}} \right) \right) = \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \psi_+ e^{-t \Delta_{\text{abs}, h}} \right) \right),$$

$$\text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \psi_- e^{-t \Delta_{\mathcal{B}, h}} \right) \right) = \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \psi_- e^{-t \Delta_{\text{rel}, h}} \right) \right).$$

**Theorem 1.** *For  $(M, \partial_+ M, \partial_- M)$ , consider the Hermitian boundary value problem  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E, g, h}$  with its corresponding  $L^2$ -realization  $\Delta_{\mathcal{B}, h}$ . If  $\phi$ ,  $\xi$  and  $\Psi$  are as in Proposition 4, then*

$$\begin{aligned} \text{LIM}_{t \rightarrow 0} \left( \text{STr} \left( \phi e^{-t \Delta_{\mathcal{B}, h}} \right) \right) &= \int_M \text{Tr}(\phi) \mathbf{e}(M, g) + (-1)^{m-1} \int_{\partial_+ M} \text{Tr}(\phi) i_+^* \mathbf{e}_{\mathbf{b}}(\partial M, g) \\ &\quad - \int_{\partial_- M} \text{Tr}(\phi) i_-^* \mathbf{e}_{\mathbf{b}}(\partial M, g). \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow 0} \left( \text{STr} \left( \Psi e^{-t \Delta_{\mathcal{B}, h}} \right) \right) = & -2 \int_M \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\mathbf{e}}(M, g, g + \tau g \xi) \wedge \omega(\nabla^E, h) \\ & -2 \int_{\partial_+ M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_+^* \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g, g + \tau g \xi) \wedge \omega(\nabla^E, h) \\ & + \text{rank}(E) \int_{\partial_+ M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_+^* B(\partial M, g + \tau g \xi) \\ & -2(-1)^m \int_{\partial_- M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_-^* \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g, g + \tau g \xi) \wedge \omega(\nabla^E, h) \\ & + (-1)^{m+1} \text{rank}(E) \int_{\partial_- M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_-^* B(\partial M, g + \tau g \xi). \end{aligned}$$

*Proof.* This follows from the results by Brüning and Ma in [BM06], stated in terms of Proposition 4 above, Proposition 5 and Lemma 8. More recently, Brüning and Ma gave also a proof of this statement, see Theorem 3.2 in [BM11], based on the methods developed in [BM06].  $\square$

**2.3. Involutions, bilinear and Hermitian forms.** We fix a Hermitian structure compatible with the bilinear as follows. Since  $E$  is endowed with a bilinear form  $b$ , there exists an anti-linear involution  $\nu$  on  $E$  satisfying

$$(21) \quad \overline{b(\nu e_1, \nu e_2)} = b(e_1, e_2) \text{ and } b(\nu e, e) > 0 \text{ for all } e_1, e_2, e \in E \text{ with } e \neq 0,$$

see for instance the proof of Theorem 5.10 in [BH07]. In this way, we obtain a (positive definite) Hermitian form on  $E$  given by

$$(22) \quad h(e_1, e_2) := b(e_1, \nu e_2).$$

Remark that  $\nabla^E \nu = 0$  is not required so that

$$h^{-1}(\nabla^E h) = \nu^{-1}(b^{-1}(\nabla^E b))\nu + \nu^{-1}(\nabla^E \nu);$$

Therefore, we end up with a Hermitian form on  $\Omega(M; E)$  compatible with  $\beta_{g,b}$ , given by  $\ll v, w \gg_{g,h} = \beta_{g,b}(v, \nu w)$ . for  $v, w \in \Omega(M; E)$ . In [SZ08] and [Su], given a bilinear form  $b$ , this involution has been exploited to study the bilinear Laplacian in terms of the Hermitian one associated to the compatible Hermitian form in (22), in both cases with and without boundary. However, our approach is a little different since we do not use a Hermitian form globally compatible with  $\beta_{g,b}$  on  $\Omega(M; E)$ , but instead a local compatibility only, see section 2.4 below. We now see what happens with the extra condition for  $\nu$  to be parallel with respect to  $\nabla^E$ .

**Lemma 9.** *Let us consider  $(M, \partial_+ M, \partial_- M)$  the compact Riemannian bordism together with the complex flat vector bundle  $E$  as above. Suppose  $E$  admits a nondegenerate symmetric bilinear form. Moreover, suppose there exists a complex anti-linear involution  $\nu$  on  $E$ , satisfying the conditions in (21) and  $\nabla^E \nu = 0$ . Let  $h$  be the (positive definite) Hermitian form on  $E$  compatible with  $b$  defined by (22). Then,  $\Delta_{E,g,b} = \Delta_{E,g,h}$  and  $\mathcal{B}_{E,g,b} = \mathcal{B}_{E,g,h}$ .*

*Proof.* Consider  $\ll \cdot, \cdot \gg_{g,h}$  the Hermitian product on  $\Omega(M; E)$ , compatible with the bilinear form, and  $d_{E,g,h}^*$ , the formal adjoint to  $d_E$  with respect to this product, which in terms of the Hodge  $\star$ -operator can be written up to a

sign as  $d_{E,g,h}^* = \pm \star_h^{-1} d_E \star_h$ . Remark that  $\nabla^E \nu = 0$  implies that  $d_E \nu = \nu d_E$ ; hence, with  $\star_h = \nu \circ \star_b$ , we have

$$(23) \quad d_{E,g,h}^* = \pm \star_h^{-1} d_E \star_h = \pm \star_b^{-1} \nu^{-1} d_E \nu \star_b = \pm \star_b^{-1} d_E \star_b = d_{E,g,b}^\sharp,$$

and therefore the Hermitian and bilinear Laplacians coincide. We turn to the assertion for the corresponding boundary operators. On the one hand, the assertion is clear for  $\mathcal{B}_{-E,g,b} = \mathcal{B}_{-E,g,h}$ , because of (23) and (5). On the other hand, for a form  $v \in \Omega^p(M; E)$  and  $\iota_{\varsigma_{\text{in}}}$ , the interior product with respect to the dual form corresponding to  $\varsigma_{\text{in}}$ , the identity  $\star_b^{\partial M} i_{\varsigma_{\text{in}}}^* v = i_{\varsigma_{\text{in}}}^* \star_b^M v$  holds; therefore the operator specifying absolute boundary can be written, independently of the Hermitian or bilinear forms, as  $\mathcal{B}_{+E,g,b}^p v = (i_{\varsigma_{\text{in}}}^* \iota_{\varsigma_{\text{in}}} v, (-1)^{p+1} i_{\varsigma_{\text{in}}}^* (d_E v)) = \mathcal{B}_{+E,g,h}^p v$ . That finishes the proof.  $\square$

**Lemma 10.** *Let  $(M, g)$  be a compact Riemannian manifold and  $E$  a flat complex vector bundle over  $M$ . Assume  $E$  is endowed with a fiber wise non-degenerate symmetric bilinear form  $b$ . For each  $x \in M$  there exists an open neighborhood  $U$  of  $x$  in  $M$ , a parallel involution  $\nu$  on  $E|_U$  and a fiberwise symmetric bilinear form  $\tilde{b}$  on  $E$  such that the family of fiber-wise symmetric bilinear forms  $b_z := b + z\tilde{b}$ , for  $z \in \mathbb{C}$ , has the following properties:*

- (i)  $b_z$  is nondegenerate for all  $z \in \mathbb{C}$  with  $|z| \leq \sqrt{2}$
- (ii)  $\overline{b_{s-i}(\nu e_1, \nu e_2)} = b_{s-i}(e_1, e_2)$ , for all  $s \in \mathbb{R}$  and  $e_i \in E|_U$ .
- (iii)  $b_{s-i}(e, \nu e) > 0$  for all  $s \in \mathbb{R}$ ,  $|s| \leq 1$  and  $0 \neq e \in E|_U$ .

*Proof.* Since flat vector bundles are locally trivial, there exists a neighborhood  $V$  of  $x$  and a parallel complex anti-linear involution  $\nu$  on  $E|_V$ . Moreover, since  $b$  is non degenerate and  $\nu$  an involution, we can assume without loss of generality that  $\nu$  can be chosen to be compatible with  $b$  at the fiber  $E_x$  over  $x$ , such that  $b_x(\nu e_1, \nu e_2) = \overline{b_x(e_1, e_2)}$  for all  $e_i \in E_x$ , and  $b_x(\nu e, e) > 0$  for all  $0 \neq e \in E_x$ . Consider

$$b^{\text{Re}}(e_1, e_2) := \frac{1}{2}(b(e_1, e_2) + \overline{b(\nu e_1, \nu e_2)}) \text{ and } b^{\text{Im}}(e_1, e_2) := \frac{1}{2i}(b(e_1, e_2) - \overline{b(\nu e_1, \nu e_2)}),$$

as symmetric bilinear forms on  $E|_V$ . In particular, note that by construction

- (a)  $b|_V = b^{\text{Re}} + i b^{\text{Im}}$  and  $b^{\text{Im}}|_{E_x} = 0$
- (b) For all  $e_i \in E|_V$ :  $\overline{b^{\text{Re}}(\nu e_1, \nu e_2)} = b^{\text{Re}}(e_1, e_2)$  and  $\overline{b^{\text{Im}}(\nu e_1, \nu e_2)} = b^{\text{Im}}(e_1, e_2)$ .

Now, choose  $\lambda : V \rightarrow [0, 1]$ , a compact supported smooth function, such that there exists an open neighborhood  $U$  of  $x$  with  $\lambda|_U = 1$ . Thus, we obtain  $\tilde{b} := \lambda b^{\text{Im}}$ , a globally defined symmetric bilinear form on  $E$ . Using  $b_{s-i}|_U = (b + (s-i)\tilde{b})|_U = b|_U + (s-i)b^{\text{Im}}|_U = b^{\text{Re}}|_U + s b^{\text{Im}}|_U$  and (b) we obtain (ii). This implies  $\overline{b_{s-i}(\nu e, e)} = b_{s-i}(\nu e, e)$ , hence  $b_{s-i}(\nu e, e)$  is real for all  $s \in \mathbb{R}$  and  $e \in E|_U$ . Finally, (i) and (iii) follow from choosing the support of  $\lambda$  sufficiently small, since  $b^{\text{Im}}|_{E_x} = 0$  and  $b_{s-i}|_{E_x}(\nu e, e) = b_x(\nu e, e) > 0$  for all  $0 \neq e \in E_x$ .  $\square$

The following Proposition is provides the key argument in the proof of Theorem 2 below.

**Proposition 6.** *Let  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E, g, b}$  be the bilinear boundary value problem under absolute and relative boundary conditions on  $(M, \partial_+ M, \partial_- M)$ . Then, for each  $x \in M$ , there exist a fiber-wise symmetric bilinear form  $\tilde{b}$  on  $E$  and a fiber-wise sesquilinear forms  $h, \tilde{h}$  on  $E$  so that the families  $b_z = b + z\tilde{b}$ , for  $z \in \mathbb{C}$  and  $h_s = h + s\tilde{h}$  for  $s \in \mathbb{R}$ , have the following properties*

- (i)  $b_z$  is nondegenerate for all  $z \in \mathbb{C}$  such that  $|z| \leq \sqrt{2}$ .
- (ii)  $h_s$  is fiber-wise Hermitian for all  $s \in \mathbb{R}$  such that  $|s| \leq 1$ .
- (iii) For each  $s \in \mathbb{R}$  with  $|s| \leq 1$ , consider  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E, g, h_s}$  the corresponding Hermitian boundary value problem. Then, there exists a neighborhood  $U$  of  $x$  such that

$$\Delta_{E, g, b_{s-i}}|_U = \Delta_{E, g, h_s}|_U \quad \text{and} \quad \mathcal{B}_{E, g, b_{s-i}}|_U = \mathcal{B}_{E, g, h_s}|_U.$$

*Proof.* By (i) in Lemma 10, for each  $x \in M$ , there exist a family  $b_z$  of nondegenerate symmetric bilinear forms on  $E$  with the required property in (i). Moreover, for each  $x \in M$ , there exist an open neighborhood  $V$  of  $x$  and a parallel complex anti-linear involution  $\nu$  on  $E|_V$ . We also know by (i) and (ii) in Lemma 10, that for each  $x \in M$  and  $|s| \leq 1$ , there exists an open neighborhood  $U \subset V$  such that  $b_{s-i}$  satisfies the conditions (i) and (ii) on  $E|_U$ . Hence,  $h_s(e_1, e_2) := b_{s-i}(\nu e_1, e_2)$  defines a Hermitian form compatible with  $b_{s-i}$  on  $E|_U$ . As in Lemma 10, we extend this form to a globally defined Hermitian form on  $E$  and obtain (ii). Then, (iii) follows from Lemma 9.  $\square$

#### 2.4. Heat trace asymptotics for *bilinear* boundary value problems.

**Lemma 11.** *For  $(M, \partial_+ M, \partial_- M)$  and  $E$  a complex flat vector bundle over  $M$ , let  $U$  be an open connected subset in  $\mathbb{C}$  and  $\{u \mapsto b_u\}_{u \in U}$  be a holomorphic family of non-degenerate symmetric bilinear forms on  $E$ . Consider  $\{(\Delta_u, \mathcal{B}_u) := (\Delta_{E, g, b_u}, \mathcal{B}_{E, g, b_u})\}_{u \in U}$  the corresponding family of bilinear boundary value problems. Then, for each  $\psi \in \text{End}(\Lambda(T^*M) \otimes E)$ , the map  $u \mapsto \text{LIM}_{t \rightarrow 0} (\text{STr}(\psi e^{-t[\Delta_{\mathcal{B}}]_u}))$  is holomorphic on  $U$ .*

*Proof.* The coefficient  $\text{LIM}_{t \rightarrow 0} (\text{STr}(\psi e^{-t[\Delta_{\mathcal{B}}]_u}))$  is computed by using (15), which requires to compute  $\mathfrak{e}_m(\Delta_u)$  and  $\mathfrak{e}_{m, k}(\Delta_u, \mathcal{B}_u)$ . By Lemma 6,  $\mathfrak{e}_m(\Delta_u)$  (resp.  $\mathfrak{e}_{m, k}(\Delta_u, \mathcal{B}_u)$ ) is locally computable as a universal polynomial belonging to the unital graded non-commutative algebra generated by germs of  $R_{i_1 i_2 i_3 i_4}, \Omega_{i_1 i_2}^u, E_u$  (resp.  $R_{i_1 i_2 i_3 i_4}, L_{ab}, \Omega_{i_1 i_2}^u, E_u, S_u, \chi_u$ ) and finite number of their covariant derivatives. Now, by definition of  $\Delta_{E, g, b}$  and  $\mathcal{B}_{E, g, b}$ , the maps  $u \mapsto \Delta_u$  and  $u \mapsto \mathcal{B}_u$  are holomorphic on  $U$ , since  $u \mapsto b_u$  is so. Therefore, the coefficients of the symbols of these operators are holomorphic functions on  $U$ . In addition, the functions  $E, S, \chi$  and their covariant derivatives are

completely determined by the symbols of  $\Delta_{E,g,b}$  and  $\mathcal{B}_{E,g,b}$  on  $U$ ; in other words, these functions depend on the polynomial coefficients of the symbols of  $\Delta_{E,g,b}$  and  $\mathcal{B}_{E,g,b}$ . Then the family  $u \mapsto (\mathbf{E}_u, \chi_u, \mathbf{S}_u)$  is holomorphic on  $U$ . This shows that  $u \mapsto \mathbf{e}_m(\Delta_u)$  and  $u \mapsto \mathbf{e}_{m,k}(\Delta_u, \mathcal{B}_u)$  are holomorphic. Finally,  $u \mapsto \text{LIM}_{t \rightarrow 0} (\text{STr}(\psi e^{-t[\Delta_{\mathcal{B}}]_u}))$  is holomorphic on  $U$ , since uniform limits on compact sets of holomorphic functions are holomorphic.  $\square$

**Theorem 2.** *For  $(M, \partial_+ M, \partial_- M)$  consider the bilinear boundary value problem  $[\Delta, \mathcal{B}]_{(M, \partial_+ M, \partial_- M)}^{E,g,b}$ , together with its  $L^2$ -realization  $\Delta_{\mathcal{B},b}$ . If  $\phi$ ,  $\xi$  and  $\Psi$  are as in Proposition 4, then*

$$(24) \quad \text{LIM}_{t \rightarrow 0} (\text{STr}(\phi e^{-t\Delta_{\mathcal{B},b}})) = \int_M \text{Tr}(\phi) \mathbf{e}(M, g) + (-1)^{m-1} \int_{\partial_+ M} \text{Tr}(\phi) i_+^* \mathbf{e}_{\mathbf{b}}(\partial M, g) - \int_{\partial_- M} \text{Tr}(\phi) \mathbf{e}_{\mathbf{b}}(\partial M, g),$$

and

$$(25) \quad \begin{aligned} \text{LIM}_{t \rightarrow 0} (\text{STr}(\Psi e^{-t\Delta_{\mathcal{B},b}})) = & -2 \int_M \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\mathbf{e}}(M, g, g + \tau g \xi) \wedge \omega(\nabla^E, b) \\ & - 2 \int_{\partial_+ M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_+^* \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g, g + \tau g \xi) \wedge \omega(\nabla^E, b) \\ & + \text{rank}(E) \int_{\partial_+ M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_+^* B(\partial M, g + \tau g \xi) \\ & - 2(-1)^m \int_{\partial_- M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_-^* \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g, g + \tau g \xi) \wedge \omega(\nabla^E, b) \\ & + (-1)^{m+1} \text{rank}(E) \int_{\partial_- M} \frac{\partial}{\partial \tau} \Big|_{\tau=0} i_-^* B(\partial M, g + \tau g \xi). \end{aligned}$$

*Proof.* By compactness of  $M$ , it suffices to show that each point  $x \in M$  admits a neighborhood  $U$  so that the formulas above hold for all  $\phi$  with  $\text{supp}(\phi) \subset U$  and  $\xi$  with  $\text{supp}(\xi) \subset U$ . For each  $x \in M$ , choose  $b_z = b + z\tilde{b}$ ,  $h_s = h + s\tilde{h}$  and  $U$  as in Proposition 6, with  $\text{supp}(\phi) \subset U$  and  $\text{supp}(\xi) \subset U$ . By Proposition 6 (iii),  $\text{LIM}_{t \rightarrow 0} \text{STr}(\phi e^{-t\Delta_{\mathcal{B},b_s-i}}) = \text{LIM}_{t \rightarrow 0} \text{STr}(\phi e^{-t\Delta_{\mathcal{B},h_s}})$ , for all  $|s| \leq 1$ , for these quantities depend on the geometry over  $U$  only. From Theorem 1, we have

$$\text{LIM}_{t \rightarrow 0} \text{STr}(\phi e^{-t\Delta_{\mathcal{B},b_s-i}}) = \int_M \text{Tr}(\phi) \mathbf{e}(M, g) + (-1)^{m-1} \int_{\partial_+ M} \text{Tr}(\phi) i_+^* \mathbf{e}_{\mathbf{b}}(\partial M, g) - \int_{\partial_- M} \text{Tr}(\phi) i_-^* \mathbf{e}_{\mathbf{b}}(\partial M, g),$$

for all  $|s| \leq 1$ . Now, since the function  $z \mapsto \text{LIM}_{t \rightarrow 0} \text{STr}(\phi e^{-t\Delta_{\mathcal{B},b_z}})$  depends holomorphically on  $z$  (see Lemma 11), that the right hand side of the equality above is constant in  $z$ , and that the domain of definition for  $z$  certainly contains an accumulation point, these formulas are extended by analytically continuation to

$$\text{LIM}_{t \rightarrow 0} \text{STr}(\phi e^{-t\Delta_{\mathcal{B},b_z}}) = \int_M \text{Tr}(\phi) \mathbf{e}(M, g) + (-1)^{m-1} \int_{\partial_+ M} \text{Tr}(\phi) i_+^* \mathbf{e}_{\mathbf{b}}(\partial M, g) - \int_{\partial_- M} \text{Tr}(\phi) i_-^* \mathbf{e}_{\mathbf{b}}(\partial M, g),$$

for all  $|z| \leq \sqrt{2}$ . After setting  $z = 0$  we obtain the desired identity in (24). Similarly, using Proposition 6 (iii), we obtain  $\text{LIM}_{t \rightarrow 0} \text{STr}(\Psi e^{-t\Delta_{\mathcal{B},b_s-i}}) = \text{LIM}_{t \rightarrow 0} \text{STr}(\Psi e^{-t\Delta_{\mathcal{B},h_s}})$ , for all  $|s| \leq 1$ , for these quantities depend on the

geometry over  $U$  only. Now, the right hand side of the equality above does depend on  $z$ , but since the map  $u \mapsto \text{Tr}(b_u^{-1} \nabla^E b_u)$  is holomorphic, the identity in (25) also follows from Lemma 11 and Theorem 1, by the same procedure as for (24).  $\square$

### 3. COMPLEX-VALUED ANALYTIC TORSION ON COMPACT BORDISMS

Let  $(M, \partial_+ M, \partial_- M)$  be a Riemannian bordism and  $E$  be complex flat vector bundle over  $M$  endowed with a nondegenerate symmetric bilinear form. Consider  $\Delta_{\mathcal{B}}$  the  $L^2$ -realization of the bilinear Laplacian acting on  $E$ -valued smooth forms satisfying absolute boundary conditions on  $\partial_+ M$  and relative ones on  $\partial_- M$ .

If  $\Omega_{\Delta_{\mathcal{B}}}(0)$  is the 0-generalized eigenspace of  $\Delta_{\mathcal{B}}$ , consider the restriction of  $\beta_{g,b}$  to  $\Omega_{\Delta_{\mathcal{B}}}(0)$ ; this is a non degenerate symmetric bilinear form in view of Proposition 1. By Lemma 3.3 in section 3 of [BH07] we obtain a nondegenerate bilinear form on  $\det H(\Omega_{\Delta_{\mathcal{B}}}(0))$ , which in turn, by Proposition 3, induces a bilinear form on  $\det(H(M, \partial_- M; E))$ , which we denote by  $\tau(0)_{E,g,b}$ . Let us denote by

$$\Delta_{\mathcal{B},q}^{\zeta} := \Delta_{\mathcal{B}}|_{\Omega_{\Delta_{\mathcal{B}}}^q(M;E)(0)^{\zeta}|_{\mathcal{B}}}$$

the restriction of  $\Delta_{\mathcal{B}}$  to  $\Omega_{\Delta_{\mathcal{B}}}^q(M;E)(0)^{\zeta}|_{\mathcal{B}}$ , the space of smooth differential forms of degree  $q$  which are not in  $\Omega_{\Delta_{\mathcal{B}}}(M;E)(0)$  but satisfy boundary conditions. Lemma 2 permits to choose a non-zero Agmon angle avoiding the spectrum of  $\Delta_{\mathcal{B},q}^{\zeta}$  so that complex powers of the bilinear Laplacian can be defined. Then, the function  $s \mapsto (\Delta_{\mathcal{B},q}^{\zeta})^{-s}$  associates to each  $s \in \mathbb{C}$ , with  $\text{Re}(s) > \dim(M)/2$ , an operator of Trace class and it extends to a meromorphic function on the complex plane which is holomorphic at 0, see [Gr], [Se67] and [Se69] or more generally, for pseudo-differential boundary value problems, see Chapter 4 in [Gb]. The  $\zeta$ -regularized determinant of  $\Delta_{\mathcal{B},q}$  is defined as

$$\det'(\Delta_{\mathcal{B},q}) := \exp\left(-\left.\frac{\partial}{\partial s}\right|_{s=0} \text{Tr}((\Delta_{\mathcal{B},q}^{\zeta})^{-s})\right).$$

From Lemma 2 this determinant does not depend on the choice of the Agmon's angle. By Lemma 3.3 in [BH07] we define the complex-valued Ray–Singer torsion on the bordism  $(M, \partial_+ M, \partial_- M)$  as the bilinear form on  $\det H(M, \partial_- M; E)$  obtained as

$$\tau_{E,g,b} := \tau(0)_{E,g,b} \prod_q (\det'(\Delta_{\mathcal{B},q}))^{(-1)^q}.$$

The following generalizes the formulas obtained in [BH07] in the case without boundary and they are based on the corresponding ones for the Ray–Singer metric in [BM06]. They also coincide with the ones obtained by Su in odd

dimensions, but they do not require that the smooth variations of  $g$  and  $b$  are supported on a compactly supported in the interior of  $M$ , see [Su].

**Theorem 3. (*Anomaly formulas*)** *Let  $(M, \partial_+ M, \partial_- M)$  be a compact Riemannian bordism and  $E$  be complex flat vector bundle over  $M$ . Consider  $g_u$  a smooth one-parameter family of Riemannian metrics on  $M$  and  $b_u$  a smooth one-parameter family of a fiber wise nondegenerate symmetric bilinear forms on  $E$  and denote by  $\dot{g}_t$  and  $\dot{b}_t$  their corresponding infinitesimal variations. Let  $\tau_{E,g_u,b_u}$  the associated family of complex valued analytic torsions. Then, we have the following logarithmic derivative*

$$\frac{\partial}{\partial w} \Big|_u \left( \frac{\tau_{E,g_u,b_u}}{\tau_{E,g_u,b_u}} \right)^2 = \mathbf{E}(b_u, g_u) + \tilde{\mathbf{E}}(b_u, g_u) + \mathbf{B}(g_u),$$

where

$$\begin{aligned} \mathbf{E}(b_u, g_u) &:= \int_M \text{Tr}(b_u^{-1} \dot{b}_u) \mathbf{e}(M, g) \\ &\quad + (-1)^{m-1} \int_{\partial_+ M} \text{Tr}(b_u^{-1} \dot{b}_u) \mathbf{e}_{\mathbf{b}}(\partial M, g_u) \\ &\quad - \int_{\partial_- M} \text{Tr}(b_u^{-1} \dot{b}_u) \mathbf{e}_{\mathbf{b}}(\partial M, g_u), \\ \tilde{\mathbf{E}}(b_u, g_u) &:= -2 \int_M \frac{\partial}{\partial t} \Big|_{t=0} \tilde{\mathbf{e}}(M, g_u, g_u + t \dot{g}_u) \wedge \omega(\nabla^E, b_u) \\ &\quad - 2 \int_{\partial_+ M} \frac{\partial}{\partial t} \Big|_{t=0} i_+^* \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g_u, g_u + t \dot{g}_u) \wedge \omega(\nabla^E, b_u) \\ &\quad - 2(-1)^m \int_{\partial_- M} \frac{\partial}{\partial t} \Big|_{t=0} i_-^* \tilde{\mathbf{e}}_{\mathbf{b}}(\partial M, g_u, g_u + t \dot{g}_u) \wedge \omega(\nabla^E, b_u), \\ \mathbf{B}(g_u) &:= \text{rank}(E) \int_{\partial_+ M} \frac{\partial}{\partial t} \Big|_{t=0} i_+^* B(\partial M, g_u + t \dot{g}_u) \\ &\quad + (-1)^{m+1} \text{rank}(E) \int_{\partial_- M} \frac{\partial}{\partial t} \Big|_{t=0} i_-^* B(\partial M, g_u + t \dot{g}_u), \end{aligned}$$

where  $\omega(\nabla^E, b) := -\frac{1}{2} \text{Tr}(b^{-1} \nabla^E b)$  is the Kamber–Tondeur form, see section 2.4 in [BH07]

*Proof.* The method described in section 6 of [BH07] leading to the infinitesimal variation of the torsion in the closed situation also holds in the situation with boundary; this was also used in [Su]. Then, by formula (54) in [BH07], we know that the problem of computing this variation boils down to computing the constant term in the asymptotic expansion for the heat kernel of  $\Delta_{\mathcal{B}}$ ; this is Theorem 2.  $\square$

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